લિબર્ટી પેપરસેટ

ધોરણ 12 : ગણિત

Full Solution

સમય : 3 કલાક **અસાઈનમેન્ટ પ્રશ્નપત્ર 6**

PART A

1. (C) **2.** (A) **3.** (C) **4.** (C) **5.** (A) **6.** (C) **7.** (B) **8.** (B) **9.** (B) **10.** (A) **11.** (B) **12.** (C) **13.** (B) **14.** (C) **15.** (A) **16.** (B) **17.** (B) **18.** (B) **19.** (A) **20.** (C) **21.** (D) **22.** (B) **23.** (B) **24.** (D) **25.** (D) **26.** (C) **27.** (C) **28.** (B) **29.** (A) **30.** (C) **31.** (A) **32.** (B) **33.** (B) **34.** (A) **35.** (C) **36.** (A) **37.** (A) **38.** (B) **39.** (A) **40.** (B) **41.** (A) **42.** (A) **43.** (A) **44.** (A) **45.** (A) **46.** (C) **47.** (A) **48.** (D) **49.** (B) **50.** (A)

PART B

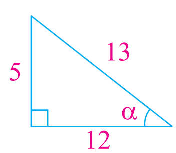
વિભાગ-A

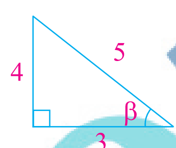
1.

જ.બા. $= sin^{-1}\frac{5}{13} + cos^{-1}\frac{3}{5}$

$sin^{-1}\frac{5}{13} = \alpha \quad , \quad cos^{-1}\frac{3}{5} = \beta$

$\therefore \quad sin\ \alpha = \frac{5}{13} \quad , \quad cos\ \beta = \frac{3}{5}$





$\therefore \quad tan\ \alpha = \frac{5}{12} \quad , \quad tan\ \beta = \frac{4}{3}$

$tan(\alpha + \beta) = \frac{tan\ \alpha + tan\ \beta}{1 - tan\ \alpha\ tan\ \beta}$

$= \frac{\frac{5}{12} + \frac{4}{3}}{1 - \frac{5}{12} \cdot \frac{4}{3}}$

$= \frac{\frac{15+48}{36}}{\frac{36-20}{36}}$

$tan(\alpha + \beta) = \frac{63}{16}$

$\therefore \quad \alpha + \beta = tan^{-1}\frac{63}{16}$

$\therefore \quad sin^{-1}\frac{5}{13} + cos^{-1}\frac{3}{5} = tan^{-1}\frac{63}{16}$

2.

$tan^{-1}\left(\frac{3a^2x - x^3}{a^3 - 3ax^2}\right)$

$= tan^{-1}\left(\frac{3\frac{x}{a} - \frac{x^3}{a^3}}{1 - 3\frac{x^2}{a^2}}\right)$

$\therefore \ \frac{x}{a} = tan\ \theta$ ધારો.

$\therefore \ \theta = tan^{-1}\frac{x}{a},\ \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$= tan^{-1}\left(\frac{3\,tan\,\theta - tan^3\theta}{1 - 3\,tan^2\theta}\right)$

$= tan^{-1}(tan\ 3\theta)$

અહીં; $\frac{-1}{\sqrt{3}} < \frac{x}{a} < \frac{1}{\sqrt{3}}$

$\therefore \ tan\left(-\frac{\pi}{6}\right) < tan\ \theta < tan\frac{\pi}{6}$

$\therefore \ -\frac{\pi}{6} < \theta < \frac{\pi}{6}$

$\therefore \ -\frac{\pi}{2} < 3\theta < \frac{\pi}{2}$ .... (1)

$= 3\ \theta$ ($\because$ પરિણામ (1) પરથી)

$= 3\ tan^{-1}\frac{x}{a}$

3.

f એ $x = 2$ આગળ સતત છે.

$\therefore \ \lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x) = f(2)$

$\therefore \ \lim_{x \to 2^+} (3) = \lim_{x \to 2^-} kx^2$

$$\begin{pmatrix} \because x \to 2^+ \\ \Rightarrow x > 2 \\ \Rightarrow f(x) = 3 \end{pmatrix} \quad \begin{pmatrix} \because x \to 2^- \\ \Rightarrow x < 2 \\ \Rightarrow f(x) = kx^2 \end{pmatrix}$$

$\therefore\ 3 = 4k$

$\therefore\ k = \frac{3}{4}$

4.

➪ $I = \int e^x \left(\frac{1+\sin x}{1+\cos x}\right) dx$

$= \int e^x \left(\frac{1 + 2\sin\frac{x}{2}\cdot\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\right) dx$

$= \int e^x \left(\frac{1}{2}\sec^2\frac{x}{2} + \tan\frac{x}{2}\right) dx$

$= \int e^x \left(\tan\frac{x}{2} + \frac{d}{dx}\left(\tan\frac{x}{2}\right)\right) dx$

$I = e^x \cdot \tan\frac{x}{2} + c \qquad \left(\because \int e^x(f(x) + f'(x))\,dx = e^x f(x) + c\right)$

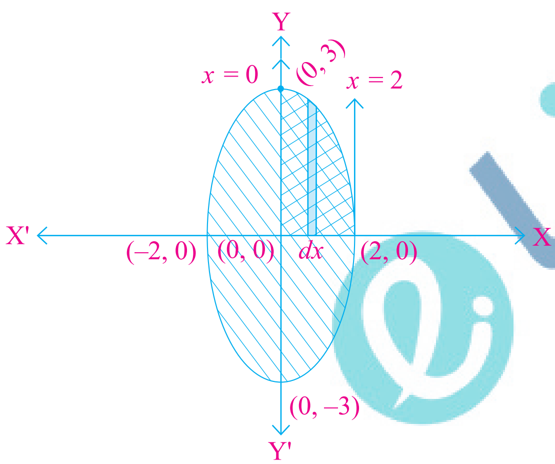
5.

➪ $\frac{x^2}{4} + \frac{y^2}{9} = 1$

$a^2 = 4,\ a = 2$

$b^2 = 9,\ b = 3$

$b > a$



➪ આવૃત્ત પ્રદેશનું ક્ષેત્રફળ :

A = 4 × પ્રથમ પ્રદેશ વડે આવૃત્ત ક્ષેત્રફળ

$\therefore A = 4|I|$

$I = \int_0^2 y\ dx$

$I = \int_0^2 \frac{3}{2}\sqrt{4-x^2}\ dx$

હવે, $\frac{x^2}{4} + \frac{y^2}{9} = 1$

$\therefore y^2 = 9\left[1 - \frac{x^2}{4}\right]$

$= \frac{9}{4}(4 - x^2)$

$\therefore y = \frac{3}{2}\sqrt{4-x^2}$

$I = \frac{3}{2}\int_0^2 \sqrt{4-x^2}\ dx$

$= \frac{3}{2}\int_0^2 \sqrt{2^2-x^2}\ dx$

$I = \frac{3}{2}\left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_0^2$

$I = \frac{3}{2}\left[\left(\frac{2}{2}(0) + 2\sin^{-1}(1)\right) - (0)\right]$

$I = \frac{3}{2}\cdot 2 \cdot \frac{\pi}{2}$

$I = \frac{3\pi}{2}$

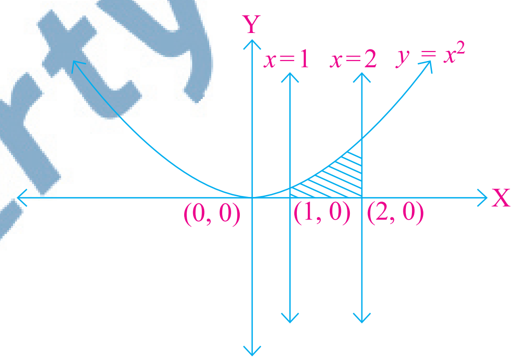
હવે, $A = 4|I|$

$= 4\left|\frac{3\pi}{2}\right|$

$\therefore\ A = 6\pi$ ચોરસ એકમ

6.

➪ $x^2 = y$



આવૃત્ત પ્રદેશનું ક્ષેત્રફળ,

$A = |I|$

$\therefore\ I = \int_1^2 y\ dx$

$\therefore\ I = \int_1^2 x^2\ dx$

$\therefore\ I = \left[\frac{x^3}{3}\right]_1^2$

$\therefore\ I = \frac{1}{3}\ ((2)^3 - (1)^3)$

$\therefore\ I = \frac{7}{3}$

હવે, $A = |I| = \left|\frac{7}{3}\right|$

$\therefore\ A = \frac{7}{3}$ ચોરસ એકમ

7.

$$y\,dx - (x + 2y^2)\,dy = 0$$

$\therefore\ y\,dx = (x + 2y^2)\,dy$

$\therefore\ \dfrac{dx}{dy} = \dfrac{x}{y} + 2y$

$\therefore\ \dfrac{dx}{dy} - \dfrac{x}{y} = 2y$

$\therefore\ P(y) = -\dfrac{1}{y},\ Q(y) = 2y$

$\text{I.F.} = e^{\int P(y)\,dy}$

$= e^{-\int \frac{1}{y}\,dy}$

$= e^{-\log y}$

$= e^{\log_e y^{-1}}$

$= \dfrac{1}{y}$

$x \cdot (\text{I.F.}) = \int Q(y)\,(\text{I.F.})\,dy$

$x \cdot \dfrac{1}{y} = \int 2y \cdot \dfrac{1}{y}\,dy$

$= 2\int 1\,dy$

$\dfrac{x}{y} = 2y + c$

$\therefore\ x = 2y^2 + cy$

જે આપેલ વિકલ સમીકરણનો વ્યાપક ઉકેલ છે.

8.

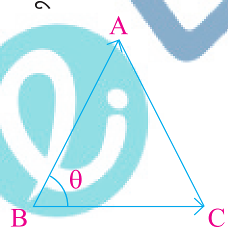
A નો સ્થાન સદિશ $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$

B નો સ્થાન સદિશ $\vec{b} = -\hat{i}$

C નો સ્થાન સદિશ $\vec{c} = 0\hat{i} + \hat{j} + 2\hat{k}$

$\angle ABC = \overrightarrow{BA}$ અને $\overrightarrow{BC}$ વચ્ચેનો ખૂણો

આ ખૂણો θ લેતાં,



$\cos\theta = \dfrac{(\overrightarrow{BA})\cdot(\overrightarrow{BC})}{|\overrightarrow{BA}| \cdot |\overrightarrow{BC}|}$

હવે, $\overrightarrow{BA} = \vec{a} - \vec{b}$

$= 2\hat{i} + 2\hat{j} + 3\hat{k}$

$|\overrightarrow{BA}| = \sqrt{4+4+9}$

$= \sqrt{17}$

તથા $\overrightarrow{BC} = \vec{c} - \vec{b}$

$= \hat{i} + \hat{j} + 2\hat{k}$

$|\overrightarrow{BC}| = \sqrt{1+1+4}$

$= \sqrt{6}$

$\overrightarrow{BA}\cdot\overrightarrow{BC} = (2\hat{i} + 2\hat{j} + 3\hat{k})(\hat{i} + \hat{j} + 2\hat{k})$

$= (2)(1) + (2)(1) + (3)(2)$

$= 2 + 2 + 6$

$= 10$

$\therefore\ \cos\theta = \dfrac{10}{\sqrt{17}\sqrt{6}}$

$= \dfrac{10}{\sqrt{102}}$

$\therefore\ \theta = \cos^{-1}\left(\dfrac{10}{\sqrt{102}}\right)$

9.

$\dfrac{x-5}{7} = \dfrac{y+2}{-5} = \dfrac{z}{1} \Rightarrow \dfrac{x-5}{7} = \dfrac{y+2}{5} = \dfrac{z-0}{1}$

$L\ :\ \vec{r} = (5\hat{i} - 2\hat{j} + 0\hat{k}) + \lambda(7\hat{i} - 5\hat{j} + \hat{k})$

$\therefore\ \vec{b_1} = 7\hat{i} - 5\hat{j} + \hat{k}$

$\dfrac{x}{1} = \dfrac{y}{2} = \dfrac{z}{3}$

$M\ :\ \vec{r} = (0\hat{i} + 0\hat{j} + 0\hat{k}) + \mu(\hat{i} + 2\hat{j} + 3\hat{k})$

$\therefore\ \vec{b_2} = \hat{i} + 2\hat{j} + 3\hat{k}$

હવે, $\vec{b_1}\cdot\vec{b_2}$

$= (7\hat{i} - 5\hat{j} + \hat{k})\cdot(\hat{i} + 2\hat{j} + 3\hat{k})$

$= 7 - 10 + 3$

$= 0$

∴ L અને M પરસ્પર લંબ છે.

10.

ધારો કે આપેલ રેખાઓના દિક્‌ગુણોત્તર,

$a_1, b_1, c_1 = a, b, c$ તથા

$a_2, b_2, c_2 = b - c, c - a, a - b$

બે રેખાઓ વચ્ચેનો ખૂણો θ લેતાં,

$\therefore\ \cos\theta = \dfrac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$

$= \dfrac{|(a,b,c)\cdot(b-c, c-a, a-b)|}{\sqrt{a^2+b^2+c^2}\sqrt{(b-c)^2+(c-a)^2+(a-b)^2}}$

$= \dfrac{|a(b-c)+b(c-a)+c(a-b)|}{\sqrt{1}\sqrt{1}}$

$= \dfrac{|ab-ac+bc-ab+ca-bc|}{1}$

$\therefore\ \cos\theta = 0$

$\therefore\ \theta = \dfrac{\pi}{2}$

આપેલ બે રેખાઓ વચ્ચેનો ખૂણો $\dfrac{\pi}{2}$ મળે.

11.

$n = 6$ S = {1, 2, 3, 4, 5, 6}

લાલ રંગથી લખેલા પૂર્ણાંક 1, 2, 3

લીલા રંગથી લખેલા પૂર્ણાંક 4, 5, 6

ઘટના A : પાસા પર મળતો પૂર્ણાંક યુગ્મ છે.

A = {2, 4, 6}

$\therefore\ r = 3$

$\therefore$ P(A) $= \dfrac{3}{6}$

$= \dfrac{1}{2}$

ઘટના B : પાસો ફેંકતા તેના પર લાલ રંગનો પૂર્ણાંક

B = {1, 2, 3}

$\therefore r = 3$

$\therefore$ P(B) $= \dfrac{3}{6} = \dfrac{1}{2}$

$\therefore A \cap B = \{2\}$

$\therefore r = 1$

$\therefore P(A \cap B) = \dfrac{1}{6}$ .... (i)

$\therefore P(A) \cdot P(B) = \dfrac{1}{2} \times \dfrac{1}{2}$

$= \dfrac{1}{4}$ .... (ii)

$\therefore P(A) \cdot P(B) \neq P(A \cap B)$

$\therefore$ A અને B પરસ્પર નિરપેક્ષ ઘટનાઓ નથી.

12.

કુટુંબના ફોટા માટે માતા-પિતા અને પુત્ર યાદૃચ્છિક રીતે એકસાથે હારમાં ઊભા રહે છે.

ધારો કે માતા $\rightarrow$ M

પિતા $\rightarrow$ F

પુત્ર $\rightarrow$ S

$\therefore$ ત્રણ વ્યક્તિઓને હારમાં ગોઠવવાના પ્રકાર $= {}_3P_3$

$\therefore$ શક્ય પરિણામો $= {}_3P_3 = 3! = 6$

$\therefore$ S = {(M, F, S), (M, S, F), (F, M, S), (F, S, M), (S, M, F), (S, F, M)}

ઘટના E : પુત્ર એક છેડા પર છે.

E = {(M, F, S), (F, M, S), (S, M, F), (S, F, M)}

$\therefore r = 4$

$\therefore P(E) = \dfrac{r}{n}$

$= \dfrac{4}{6}$

$= \dfrac{2}{3}$

ઘટના F : પિતા મધ્યમાં છે.

F : {(M, F, S), (S, F, M)}

$\therefore r = 2$

$\therefore P(F) = \dfrac{2}{6} = \dfrac{1}{3}$

$E \cap F$ = {(M, F, S), (S, F, M)}

$\therefore r = 2$

$\therefore P(E \cap F) = \dfrac{1}{3}$

$\therefore P(E \mid F) = \dfrac{P(E \cap F)}{P(F)}$

$= \dfrac{\frac{1}{3}}{\frac{1}{3}}$

$= 1$

વિભાગ-B

13.

અહીં $f : N \rightarrow N,\ f(n) = \begin{cases} \dfrac{n+1}{2} & n \text{ અયુગ્મ} \\ \dfrac{n}{2} & n \text{ યુગ્મ,} \end{cases}$

$n_1 = 3, n_2 = 4$ લેતાં,

$f(n_1) = \dfrac{3+1}{2}$ તથા $f(n_2) = f(4)$

$= 2$ $\qquad = \dfrac{4}{2} = 2$

અહીં $n_1 \neq n_2$ પરંતુ $f(n_1) = f(n_2)$

$\therefore$ f એ એક-એક વિધેય નથી.

પ્રદેશ N = {1, 2, 3, 4, 5, 6,......}

$f(n) = \begin{cases} \dfrac{n+1}{2} & n \text{ અયુગ્મ} \\ \dfrac{n}{2} & n \text{ યુગ્મ,} \end{cases}$

$f(1) = \dfrac{1+1}{2} = 1$

$f(2) = \dfrac{2}{2} = 1$

$f(3) = \dfrac{3+1}{2} = 2$

$f(4) = \dfrac{4}{2} = 2$

$f(5) = \dfrac{5+1}{2} = 3$

$f(6) = \dfrac{6}{2} = 3$

$\therefore R_f = \{1, 2, 3, 4 ....\} = N$ (સહપ્રદેશ)

$\therefore$ f વ્યાપ્ત વિધેય છે.

14.

I એ 2 કક્ષાવાળો એકમ શ્રેણિક છે.

$\therefore I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

હવે, $I + A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 0 \end{bmatrix}$

$I + A = \begin{bmatrix} 1 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 1 \end{bmatrix}$ .... (1)

હવે, $(I - A) \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

$= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 0 \end{bmatrix} \right\} \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

$$= \begin{bmatrix} 1 & tan\frac{\alpha}{2} \\ -tan\frac{\alpha}{2} & 1 \end{bmatrix} \begin{bmatrix} cos\,\alpha & -sin\,\alpha \\ sin\,\alpha & cos\,\alpha \end{bmatrix}$$

$$= \begin{bmatrix} cos\,\alpha + sin\,\alpha \cdot tan\frac{\alpha}{2} & -sin\,\alpha + cos\,\alpha \cdot tan\frac{\alpha}{2} \\ -cos\,\alpha\, tan\frac{\alpha}{2} + sin\,\alpha & sin\,\alpha\, tan\frac{\alpha}{2} + cos\,\alpha \end{bmatrix}$$

હવે, $cos\ \alpha + sin\ \alpha\ tan\frac{\alpha}{2}$

$$= cos\ \alpha + 2\ sin\frac{\alpha}{2}\ cos\frac{\alpha}{2} \cdot \frac{sin\frac{\alpha}{2}}{cos\frac{\alpha}{2}}$$

$\left(\because sin\ 2\theta = 2\ sin\ \theta\ cos\ \theta\right.$ અને $\left. tan\ \theta = \frac{sin\,\theta}{cos\,\theta}\right)$

$$= cos\ \alpha + 2sin^2\frac{\alpha}{2}$$

$$= cos\ \alpha + 1 - cos\ \alpha$$

$$= 1$$

$- sin\ \alpha + cos\ \alpha \cdot tan\frac{\alpha}{2}$

$$= -2sin\frac{\alpha}{2}\ cos\frac{\alpha}{2} + cos\alpha \cdot \frac{sin\frac{\alpha}{2}}{cos\frac{\alpha}{2}}$$

$$= sin\frac{\alpha}{2}\left[-2cos\frac{\alpha}{2} + \frac{cos\,\alpha}{cos\frac{\alpha}{2}}\right]$$

$$= \frac{sin\frac{\alpha}{2}}{cos\frac{\alpha}{2}}\left[-2\,cos^2\frac{\alpha}{2} + cos\,\alpha\right]$$

$$= tan\frac{\alpha}{2}\left[-2cos^2\frac{\alpha}{2} + 2cos^2\frac{\alpha}{2} - 1\right]$$

$$= -tan\frac{\alpha}{2}$$

$$\therefore (I - A)\begin{bmatrix} cos\,\alpha & -sin\,\alpha \\ sin\,\alpha & cos\,\alpha \end{bmatrix} = \begin{bmatrix} 1 & -tan\frac{\alpha}{2} \\ tan\frac{\alpha}{2} & 1 \end{bmatrix} \quad ...(2)$$

(1) અને (2) પરથી

$$I + A = (I - A)\begin{bmatrix} cos\,\alpha & -sin\,\alpha \\ sin\,\alpha & cos\,\alpha \end{bmatrix}$$

15.

➪ ધારો કે, $u = x^{xcosx}$ અને $v = \frac{x^2+1}{x^2-1}$

$\therefore y = u + v$

હવે, બંને બાજુ x પ્રત્યે વિકલન કરતાં,

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad ......\ (1)$$

અહીં, $u = x^{xcosx}$ ની

બંને બાજુ log લેતાં,

$log\ u = xcosx \cdot logx$

હવે, બંને બાજુ x પ્રત્યે વિકલન કરતાં,

$$\therefore \frac{1}{u}\frac{du}{dx} = x \cdot cosx\ \frac{d}{dx}\ logx + cosx \cdot logx\ \frac{d}{dx}\ x + x \cdot logx\ \frac{d}{dx}\ cosx$$

$$\therefore \frac{1}{u}\frac{du}{dx} = x \cdot cosx \cdot \frac{1}{x} + cosx \cdot logx - xlogx\ sinx$$

$$\therefore \frac{du}{dx} = u\ [cosx + cosx\ logx - xlogx\ sinx]$$

$$\therefore \frac{du}{dx} = x^{xcosx}\ [cosx + cosx\ logx - xlogx\ sinx] \quad ......\ (2)$$

હવે, $v = \frac{x^2+1}{x^2-1}$ ની

બંને બાજુ x પ્રત્યે વિકલન કરતાં,

$$\frac{dv}{dx} = \frac{(x^2-1)\frac{d}{dx}(x^2+1) - (x^2+1)\frac{d}{dx}(x^2-1)}{(x^2-1)^2}$$

$$= \frac{(x^2-1)(2x+0) - (x^2+1)(2x-0)}{(x^2-1)^2}$$

$$= \frac{2x^3 - 2x - 2x^3 - 2x}{(x^2-1)^2}$$

$$\therefore \frac{dv}{dx} = \frac{-4x}{(x^2-1)^2} \quad ......\ (3)$$

પરિણામ (1) માં પરિણામ (2) અને (3) ની કિંમત મૂકતાં,

$$\frac{dy}{dx} = x^{xcosx}[cosx + cosx\ logx - xlogx\ sinx] - \frac{4x}{(x^2-1)^2}$$

16.

➪ અહીં, $f(x) = 12x^{\frac{4}{3}} - 6x^{\frac{1}{3}}$

$$\therefore f'(x) = 16x^{\frac{1}{3}} - \frac{2}{x^{\frac{2}{3}}}$$

$$= \frac{2(8x-1)}{x^{\frac{2}{3}}}$$

આથી, $f'(x) = 0$ લેતાં, $x = \frac{1}{8}$ મળે.

વધુમાં $x = 0$ આગળ $f'(x)$ વ્યાખ્યાયિત નથી.

આથી, $x = 0$ અને $x = \frac{1}{8}$ નિર્ણાયક સંખ્યાઓ/ બિંદુઓ છે.

હવે, આ નિર્ણાયક સંખ્યાઓ તથા અંતરાલનાં અંત્યબિંદુઓ $x = -1$ તથા $x = 1$ આગળ વિધેય f નાં મૂલ્યો મેળવીએ.

$$\therefore f(-1) = 12(-1)^{\frac{4}{3}} - 6(-1)^{\frac{1}{3}}$$

$$= 18$$

$$f(0) = 12(0) - 6(0)$$

$$= 0$$

$$f\left(\frac{1}{8}\right) = 12\left(\frac{1}{8}\right)^{\frac{4}{3}} - 6\left(\frac{1}{8}\right)^{\frac{1}{3}}$$

$= \frac{-9}{4}$ તથા

$$f(1) = 12(1)^{\frac{4}{3}} - 6(1)^{\frac{1}{3}}$$

$$= 6$$

આથી, આપણે કહી શકીએ કે,

વિધેય f ને $x = -1$ આગળ

વૈશ્વિક મહત્તમ મૂલ્ય 18 તથા $x = \frac{1}{8}$ આગળ

વૈશ્વિક ન્યૂનતમ મૂલ્ય $\frac{-9}{4}$ મળે છે.

17.

અહીં, $|\vec{a}| = |\vec{b}| = |\vec{c}| = k$ ધારો ... (1)

$(k > 0)$

$\vec{a} \perp \vec{b}$; $\vec{b} \perp \vec{c}$; $\vec{c} \perp \vec{a}$

$\therefore$ $\vec{a} \cdot \vec{b} = 0$; $\vec{b} \cdot \vec{c} = 0$; $\vec{c} \cdot \vec{a} = 0$ ... (2)

$$|\vec{a} + \vec{b} + \vec{c}|^2$$
$$= |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2\vec{a} \cdot \vec{b} + 2\vec{b} \cdot \vec{c} + 2\vec{c} \cdot \vec{a}$$
$$= k^2 + k^2 + k^2 + 0 + 0 + 0$$

[$\because$ પરિણામ (1), (2)]

$$|\vec{a} + \vec{b} + \vec{c}|^2 = 3k^2$$
$$\therefore \quad |\vec{a} + \vec{b} + \vec{c}| = \sqrt{3}\,k$$

ધારો કે $\vec{a} + \vec{b} + \vec{c}$ એ $\vec{a}$ સાથે α માપનો ખૂણો બનાવે છે.

$$\therefore \cos\alpha = \frac{(\vec{a} + \vec{b} + \vec{c}) \cdot \vec{a}}{|\vec{a} + \vec{b} + \vec{c}||\vec{a}|}$$
$$= \frac{\vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{a} + \vec{c} \cdot \vec{a}}{\sqrt{3}\,k \cdot k}$$
$$= \frac{|\vec{a}|^2 + 0 + 0}{\sqrt{3}\,k^2}$$
$$\therefore \cos\alpha = \frac{k^2}{\sqrt{3}\,k^2}$$
$$\therefore \cos\alpha = \frac{1}{\sqrt{3}}$$

ધારો કે $\vec{a} + \vec{b} + \vec{c}$ એ $\vec{b}$ સાથે β માપનો ખૂણો બનાવે છે.

$$\therefore \cos\beta = \frac{(\vec{a} + \vec{b} + \vec{c}) \cdot \vec{b}}{|\vec{a} + \vec{b} + \vec{c}||\vec{b}|}$$
$$= \frac{\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{b} + \vec{c} \cdot \vec{b}}{\sqrt{3}\,k \cdot k}$$
$$\therefore \cos\beta = \frac{k^2}{\sqrt{3}\,k^2}$$
$$\therefore \cos\beta = \frac{1}{\sqrt{3}}$$

ધારો કે $\vec{a} + \vec{b} + \vec{c}$ એ $\vec{c}$ સાથે γ માપનો ખૂણો બનાવે છે.

$$\therefore \cos\gamma = \frac{(\vec{a} + \vec{b} + \vec{c}) \cdot \vec{c}}{|\vec{a} + \vec{b} + \vec{c}||\vec{c}|}$$
$$= \frac{\vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{c}}{|\vec{a} + \vec{b} + \vec{c}||\vec{c}|}$$
$$= \frac{k^2}{\sqrt{3}\,k^2}$$
$$\therefore \cos\gamma = \frac{1}{\sqrt{3}}$$

$$\therefore \cos\alpha = \cos\beta = \cos\gamma = \frac{1}{\sqrt{3}}$$

$\therefore$ $\vec{a} + \vec{b} + \vec{c}$ એ $\vec{a}$, $\vec{b}$ અને $\vec{c}$ સાથે સમાન માપના ખૂણા બનાવે છે.

18.

બે રેખાઓ સમાંતર છે.

આપણી પાસે $\vec{a_1} = \hat{i} + 2\hat{j} - 4\hat{k}$,

$\vec{a_2} = 3\hat{i} + 3\hat{j} - 5\hat{k}$ અને

$\vec{b} = 2\hat{i} + 3\hat{j} + 6\hat{k}$ છે.

આથી, રેખાઓ વચ્ચેનું અંતર

$$d = \left|\frac{\vec{b} \times (\vec{a_2} - \vec{a_1})}{|\vec{b}|}\right|$$
$$= \left|\frac{\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 6 \\ 2 & 1 & -1 \end{vmatrix}}{\sqrt{4+9+36}}\right|$$
$$= \frac{|-9\hat{i} + 14\hat{j} - 4\hat{k}|}{\sqrt{49}}$$
$$= \frac{\sqrt{293}}{\sqrt{49}}$$
$$= \frac{\sqrt{293}}{7} \text{ એકમ}$$

19.

$x + 3y \geq 3$

$x + y \geq 2$

હેતુલક્ષી વિધેય $Z = 3x + 5y$

$x \geq 0$

$y \geq 0$

$x + 3y = 3$ ... (i)

x	0	3
y	1	0

$x + y = 2$ ... (ii)

x	0	2
y	2	0

$\left(\frac{3}{2}, \frac{1}{2}\right)$

(0, 0)

(i) અને (ii)નો ઉકેલ,

$\therefore 3 - 3y = 2 - y$

$\therefore 2y = 1$

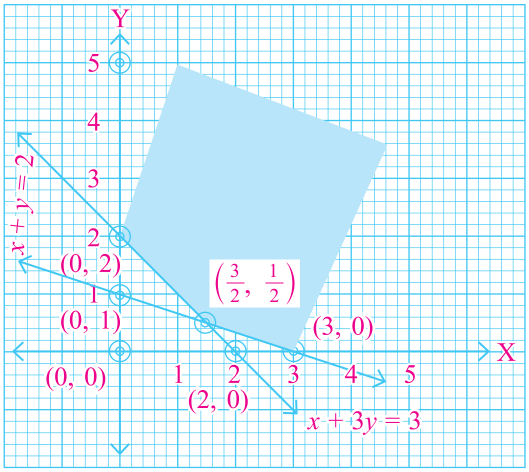
$\therefore y = \frac{1}{2}$

$y = \frac{1}{2}$ ને સમી (i)માં મૂકતાં

$\therefore x + 3\left(\frac{1}{2}\right) = 3$

$\therefore x = 3 - \frac{3}{2}$

$x = \frac{3}{2}$



આકૃતિમાં આપેલ અસમતાઓનો આલેખ દર્શાવ્યો છે જે સિમિત છે. શક્ય ઉકેલપ્રદેશનાં શિરોબિંદુઓ (0, 2), $\left(\frac{3}{2}, \frac{1}{2}\right)$ અને (3, 0) મળે.

શક્ય ઉકેલ પ્રદેશના શિરોબિંદુ	$Z = 3x + 5y$
(0, 2)	10
(3, 0)	9
$\left(\frac{3}{2}, \frac{1}{2}\right)$	7 ← ન્યૂનતમ

આમ, બિંદુ $\left(\frac{3}{2}, \frac{1}{2}\right)$ આગળ ન્યૂનતમ મૂલ્ય 7 મળે.

20.

➪ ઘટના E_1 : પસંદ કરેલ દડો પહેલા થેલામાંનો હોય

ઘટના E_2 : પસંદ કરેલ દડો બીજા થેલામાંનો હોય

ઘટના A : પસંદ થયેલ બીજો દડો લાલ રંગનો હોય

$\therefore \quad P(E_1 \mid A) = \dfrac{P(E_1) \cdot P(A \mid E_1)}{P(A)}$ (બેયઝ નિયમ)

$P(E_1) = \dfrac{1}{2} \quad ; \quad P(E_2) = \dfrac{1}{2}$

$P(A \mid E_1) = \dfrac{{}^4C_1}{{}^8C_1} = \dfrac{4}{8} = \dfrac{1}{2}$

$P(A \mid E_2) = \dfrac{{}^2C_1}{{}^8C_1} = \dfrac{2}{8} = \dfrac{1}{4}$

$\therefore \quad P(A) = P(E_1) \cdot P(A \mid E_1) + P(E_2) \cdot P(A \mid E_2)$

$= \dfrac{1}{2} \times \dfrac{1}{2} + \dfrac{1}{2} \times \dfrac{1}{4}$

$= \dfrac{1}{4} + \dfrac{1}{8}$

$= \dfrac{3}{8}$

આમ, પસંદ થયેલ દડો લાલ રંગનો અને તે પ્રથમ થેલામાંથી હોય તેની સંભાવના,

$\therefore \; P(E_1 \mid A) = \dfrac{P(A \mid E_1) \cdot P(E_1)}{P(A)}$

$= \dfrac{\frac{1}{2} \times \frac{1}{2}}{\frac{3}{8}}$

$= \dfrac{2}{3}$

21.

➪ $A^2 = A \cdot A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$

$= \begin{bmatrix} 1+1+2 & 1+2-1 & 1-3+3 \\ 1+2-6 & 1+4+3 & 1-6-9 \\ 2-1+6 & 2-2-3 & 2+3+9 \end{bmatrix}$

$A^2 = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix}$

$A^3 = A^2 \cdot A = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$

$= \begin{bmatrix} 4+2+2 & 4+4-1 & 4-6+3 \\ -3+8-28 & -3+16+14 & -3-24-42 \\ 7-3+28 & 7-6-14 & 7+9+42 \end{bmatrix}$

$A^3 = \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix}$

ડા.બા. $= A^3 - 6A^2 + 5A + 11\,I$.

$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} - 6\begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} + 5\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} + 11\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} + \begin{bmatrix} -24 & -12 & -6 \\ 18 & -48 & 84 \\ -42 & 18 & -84 \end{bmatrix} + \begin{bmatrix} 5 & 5 & 5 \\ 5 & 10 & -15 \\ 10 & -5 & 15 \end{bmatrix} + \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$

$= \begin{bmatrix} 8-24+5+11 & 7-12+5+0 & 1-6+5+0 \\ -23+18+5+0 & 27-48+10+11 & -69+84-15+0 \\ 32-42+10+0 & -13+18-5+0 & 58-84+15+11 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O =$ જ.બા.

$A^3 - 6A^2 + 5A + 11I = O$

બંને બાજુ A^{-1} વડે ગુણતાં,

$\therefore \; (A^3)A^{-1} - 6(A^2)A^{-1} + 5AA^{-1} + 11IA^{-1} = OA^{-1}$

$\therefore \; A^2 - 6A + 5I + 11A^{-1} = O$

$\therefore \; 11A^{-1} = 6A - A^2 - 5I$

$\therefore \; 11A^{-1} = 6\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\therefore\ 11A^{-1} = \begin{bmatrix} 6 & 6 & 6 \\ 6 & 12 & -18 \\ 12 & -6 & 18 \end{bmatrix} + \begin{bmatrix} -4 & -2 & -1 \\ 3 & -8 & 14 \\ -7 & 3 & -14 \end{bmatrix} + \begin{bmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

$$\therefore\ 11A^{-1} = \begin{bmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{bmatrix}$$

$$\therefore\ A^{-1} = \frac{1}{11}\begin{bmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{bmatrix}$$

$$\therefore\ A^{-1} = \begin{bmatrix} -\frac{3}{11} & \frac{4}{11} & \frac{5}{11} \\ \frac{9}{11} & -\frac{1}{11} & -\frac{4}{11} \\ \frac{5}{11} & -\frac{3}{11} & -\frac{1}{11} \end{bmatrix}$$

વિભાગ-C

22.

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

હવે,

$$A^2 = A \cdot A$$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta - \sin^2\theta & \sin\theta\cos\theta + \sin\theta\cos\theta \\ -\sin\theta\cos\theta & -\sin^2\theta + \cos^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2\theta & 2\sin\theta\cos\theta \\ -2\sin\theta\cos\theta & \cos 2\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix}$$

$$A^3 = A^2 \cdot A$$

$$= \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2\theta\cos\theta - \sin 2\theta\sin\theta & \cos 2\theta\sin\theta + \sin 2\theta\cos\theta \\ -\sin 2\theta\cos\theta - \cos 2\theta\sin\theta & -\sin 2\theta\sin\theta + \cos 2\theta\cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos 3\theta & \sin 3\theta \\ -\sin 3\theta & \cos 3\theta \end{bmatrix}$$

તે જ રીતે,

$$A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$$

23.

$$A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{vmatrix}$$

$= 2(-4 + 4) + 3(-6 + 4) + 5(3 - 2)$

$= 0 + 3(-2) + 5(1)$

$= -6 + 5$

$= -1 \neq 0$

$\therefore$ A^{-1} નું અસ્તિત્વ છે.

adj A મેળવવા માટે,

2 નો સહઅવયવ $A_{11} = (-1)^2 \begin{vmatrix} 2 & -4 \\ 1 & -2 \end{vmatrix}$

$= 1(-4 + 4)$

$= 0$

-3 નો સહઅવયવ $A_{12} = (-1)^3 \begin{vmatrix} 3 & -4 \\ 1 & -2 \end{vmatrix}$

$= (-1)(-6 + 4)$

$= 2$

5 નો સહઅવયવ $A_{13} = (-1)^4 \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix}$

$= 1(3 - 2)$

$= 1$

3 નો સહઅવયવ $A_{21} = (-1)^3 \begin{vmatrix} -3 & 5 \\ 1 & -2 \end{vmatrix}$

$= (-1)(6 - 5)$

$= -1$

2 નો સહઅવયવ $A_{22} = (-1)^4 \begin{vmatrix} 2 & 5 \\ 1 & -2 \end{vmatrix}$

$= 1(-4 - 5)$

$= -9$

-4 નો સહઅવયવ $A_{23} = (-1)^5 \begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix}$

$= (-1)(2 + 3)$

$= -5$

1 નો સહઅવયવ $A_{31} = (-1)^4 \begin{vmatrix} -3 & 5 \\ 2 & -4 \end{vmatrix}$

$= 1(12 - 10)$

$= 2$

1 નો સહઅવયવ $A_{32} = (-1)^5 \begin{vmatrix} 2 & 5 \\ 3 & -4 \end{vmatrix}$

$= (-1)(-8 - 15)$

$= 23$

-2 નો સહઅવયવ $A_{33} = (-1)^6 \begin{vmatrix} 2 & -3 \\ 3 & 2 \end{vmatrix}$

$= 1(4 + 9)$

$= 13$

$$\therefore\ adj\ A = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\ adj\ A$$

$$= \frac{1}{(-1)}\begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$$

$$\therefore\ A^{-1} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}$$

હવે, $2x - 3y + 5z = 11$

$3x + 2y - 4z = -5$

$x + y - 2z = -3$

➪ શ્રેણિક સ્વરૂપે લખતાં,

$$\therefore \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$

$\therefore$ AX = B

જ્યાં, $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$

$\therefore \quad X = A^{-1}B$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix} \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} 0-5+6 \\ -22-45+69 \\ -11-25+39 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

ઉકેલ : $x = 1, y = 2, z = 3$

24.

➪ જુઓ કે પ્રત્યેક વાસ્તવિક $t > 0$ માટે y અને x બંને વ્યાખ્યાયિત છે.

સ્પષ્ટ છે કે,

$$\frac{dy}{dt} = \frac{d}{dt}\left(a^{t+\frac{1}{t}}\right) = a^{t+\frac{1}{t}} \frac{d}{dt}\left(t+\frac{1}{t}\right) \cdot \log a$$

$$= a^{t+\frac{1}{t}} \left(1-\frac{1}{t^2}\right) \cdot \log a$$

આ જ રીતે, $\frac{dx}{dt} = a\left(t+\frac{1}{t}\right)^{a-1} \cdot \frac{d}{dt}\left(t+\frac{1}{t}\right)$

$$= a\left(t+\frac{1}{t}\right)^{a-1} \cdot \left(1-\frac{1}{t^2}\right)$$

જો $t \neq \pm 1$ તો અને તો જ $\frac{dx}{dt} \neq 0$.

આમ, $t \neq 1$ માટે, ($t > 0$ હોવાથી)

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{a^{t+\frac{1}{t}}\left(1-\frac{1}{t^2}\right)\log a}{a\left(t+\frac{1}{t}\right)^{a-1} \cdot \left(1-\frac{1}{t^2}\right)}$$

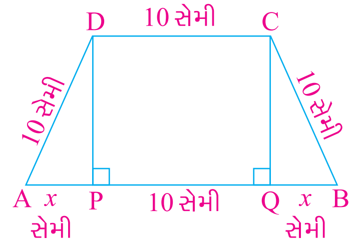
$$= \frac{a^{t+\frac{1}{t}} \log a}{a\left(t+\frac{1}{t}\right)^{a-1}}$$

$$= \frac{a^{t+\frac{1}{t}-1} \log a}{\left(t+\frac{1}{t}\right)^{a-1}} \qquad (t > 0,\ t \neq 1)$$

નોંધ : એક વિધેય $u = f(x)$ નો બીજા વિધેય $v = g(x)$ ને સાપેક્ષ વિકલિત, સંકેત $\frac{du}{dv}$ દ્વારા દર્શાવવામાં આવે છે અને તે $\frac{\frac{du}{dx}}{\frac{dv}{dx}}$ છે. જ્યાં, $\frac{dv}{dx} \neq 0$.

25.

➪ માંગેલ સમલંબ ચતુષ્કોણ આકૃતિમાં દર્શાવેલ છે.



$\overline{AB}$ પર લંબ $\overline{DP}$ તથા $\overline{CQ}$ દોરો.

ધારો કે $AP = x$ સેમી

અહીં, $\Delta APD \cong \Delta BQC$

આથી, $QB = x$ સેમી

પાયથાગોરસ પ્રમેય પરથી,

$DP = QC = \sqrt{100-x^2}$

ધારો કે, સમલંબ ચતુષ્કોણનું ક્ષેત્રફળ S છે.

$\therefore$ S $\equiv$ S(x)

$$= \frac{1}{2} \text{ (સમાંતર બાજુઓનો સરવાળો)(ઊંચાઈ)}$$

$$= \frac{1}{2}(2x + 10 + 10)\left(\sqrt{100-x^2}\right)$$

$$= (x + 10)\left(\sqrt{100-x^2}\right)$$

$$\therefore S'(x) = (x+10)\frac{(-2x)}{2\sqrt{100-x^2}} + \left(\sqrt{100-x^2}\right)$$

$$= \frac{-2x^2-10x+100}{\sqrt{100-x^2}}$$

હવે, $S'(x) = 0$ લેતાં,

$2x^2 + 10x - 100 = 0$ એટલે કે,

$x = 5$ તથા $x = -10$ મળે.

પરંતુ x એ અંતર દર્શાવે છે.

તે ઋણ ન હોઈ શકે.

આથી, $x = 5$ હવે,

$$S''(x) = \frac{\sqrt{100-x^2}(-4x-10)-(-2x^2-10x+100)\frac{(-2x)}{2\sqrt{100-x^2}}}{100-x^2}$$

$$= \frac{2x^3-300x-1000}{(100-x^2)^{\frac{3}{2}}} \qquad \text{(સાદું રૂપ આપતાં)}$$

અથવા $S''(5) = \frac{2(5)^3-300(5)-1000}{(100-(5)^2)^{\frac{3}{2}}}$

$$= \frac{-2250}{75\sqrt{75}} = \frac{-30}{\sqrt{75}} < 0$$

આથી, $x = 5$ આગળ સમલંબ ચતુષ્કોણનું ક્ષેત્રફળ મહત્તમ હોય.

$\therefore$ મહત્તમ ક્ષેત્રફળ $S(5) = (5 + 10)\sqrt{100-(5)^2}$

$$= 15\sqrt{75}$$

$$= 75\sqrt{3} \text{ સેમી}^2$$

26.

$$I = \int \frac{\sqrt{x^2+1}\,[log(x^2+1) - 2\,log\,x]}{x^4}\,dx$$

$$= \int \frac{\sqrt{x^2+1}\,[log(x^2+1) - log\,x^2]}{x^4}\,dx$$

$$= \int \frac{\sqrt{x^2+1}}{x^4}\,log\left(\frac{x^2+1}{x^2}\right)dx$$

$$= \int \frac{x\sqrt{1+\frac{1}{x^2}}}{x^4}\,log\left(1+\frac{1}{x^2}\right)dx$$

$$I = \int \frac{\sqrt{1+\frac{1}{x^2}}}{x^3}\,log\left(1+\frac{1}{x^2}\right)dx$$

હવે, $1 + \frac{1}{x^2} = t^2$ આદેશ લેતાં,

$\therefore \frac{-2}{x^3}\,dx = 2t \cdot dt$

$\therefore \frac{dx}{x^3} = -t\,dt$

$$I = \int t \cdot log(t^2)\,(-t\,dt)$$

$$= \int -2t^2 \cdot log\,t\,dt$$

$$I = -2\int t^2 \cdot log\,t\,dt$$

$$I = -2\,I_1 \quad \text{... (1)}$$

હવે, $I_1 = \int t^2 \cdot log\,t\,dt$

$u = log\,t \quad ; \quad v = t^2$

$\therefore \frac{du}{dt} = \frac{1}{t}$

$$I_1 = log\,t\int t^2\,dt - \int\left[\frac{1}{t}\int t^2\,dt\right]dt$$

$$= \frac{log\,t \cdot t^3}{3} - \int \frac{1}{t}\cdot\frac{t^3}{3}\,dt$$

$$= \frac{log\,t \cdot t^3}{3} - \frac{1}{3}\int t^2\,dt$$

$$I_1 = \frac{log\,t \cdot t^3}{3} - \frac{t^3}{9} + c$$

હવે, $1 + \frac{1}{x^2} = t^2$ હોવાથી,

$\therefore t = \sqrt{1+\frac{1}{x^2}}$ અને $t^3 = \left(1+\frac{1}{x^2}\right)^{\frac{3}{2}}$

$$I_1 = \frac{t^3}{3}\left[log\,t - \frac{1}{3}\right] + c$$

$$I_1 = \frac{1}{3}\left(1+\frac{1}{x^2}\right)^{\frac{3}{2}}\left[log\left(1+\frac{1}{x^2}\right)^{\frac{1}{2}} - \frac{1}{3}\right] + c_1$$

$$I = \frac{1}{3}\left(1+\frac{1}{x^2}\right)^{\frac{3}{2}}\left[\frac{1}{2}\,log\left[1+\frac{1}{x^2}\right] - \frac{1}{3}\right] + c_1$$

I_1 ની કિંમત પરિણામ (1) માં મૂકતાં,

$$I = \frac{-2}{3}\left[1+\frac{1}{x^2}\right]^{\frac{3}{2}}\left[\frac{1}{2}\,log\left(1+\frac{1}{x^2}\right) - \frac{1}{3}\right] + c$$

$$I = \frac{-1}{3}\left[1+\frac{1}{x^2}\right]^{\frac{3}{2}}\left[log\left(1+\frac{1}{x^2}\right) - \frac{2}{3}\right] + c$$

27.

આપેલ વિકલ સમીકરણ નીચે પ્રમાણે લખી શકાય :

$$\frac{dy}{dx} = \frac{y\cos\left(\frac{y}{x}\right) + x}{x\cos\left(\frac{y}{x}\right)} \quad \text{... (1)}$$

એ $\frac{dy}{dx} = F(x, y)$ પ્રકારનું વિકલ સમીકરણ છે.

અહીં, $F(x, y) = \frac{y\cos\left(\frac{y}{x}\right) + x}{x\cos\left(\frac{y}{x}\right)}$

x ની જગ્યાએ λx અને y ની જગ્યાએ λy મૂકતાં,

$$F(\lambda x, \lambda y) = \frac{\lambda\left[y\cos\left(\frac{y}{x}\right) + x\right]}{\lambda\left[y\cos\left(\frac{y}{x}\right)\right]}$$

$$= \lambda^0\,F(x, y)$$

આમ, $F(x, y)$ એ શૂન્ય ઘાતવાળું સમપરિમાણ વિધેય છે. માટે આપેલ સમીકરણ સમપરિમાણ વિકલ સમીકરણ છે. તેનો ઉકેલ શોધવા માટે આપણે,

$y = vx$ લઈએ. ... (2)

સમીકરણ (2) નું 'x' ને સાપેક્ષ વિકલન કરતાં,

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \quad \text{... (3)}$$

y અને $\frac{dy}{dx}$ ની કિંમતો સમીકરણ (1) માં મૂકતાં,

$$\therefore v + x\frac{dv}{dx} = \frac{v\cos v + 1}{\cos v}$$

$$\therefore x\frac{dv}{dx} = \frac{v\cos v + 1}{\cos v} - v$$

$$\therefore x\frac{dv}{dx} = \frac{1}{\cos v}$$

$$\therefore \cos v\,dv = \frac{dx}{x}$$

$$\therefore \int \cos v\,dv = \int \frac{1}{x}\,dx$$

$$\therefore \sin v = log\,|x| + log\,|c|$$

$$\therefore \sin v = log\,|cx|$$

$\rightarrow v = \frac{y}{x}$ મૂકતાં,

$$\sin\left(\frac{y}{x}\right) = log\,|cx|$$

આ વિકલ સમીકરણ (1) નો જરૂરી વ્યાપક ઉકેલ છે.